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Two noncolourable configurations in four dimensions illustrating the Kochen–Specker theorem

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Abstract. It is demonstrated that the 60 rays corresponding to antipodal pairs of vertices of the 600-cell, and the 300 rays corresponding to antipodal pairs of vertices of the 120-cell, can both be used to give noncolouring proofs of the Bell–Kochen–Specker theorem in four dimensions.

The Bell–Kochen–Specker (BKS) theorem [1, 2], which is closely related to a more powerful result proved earlier by Gleason [3], asserts that in a Hilbert space of dimension $n \geqslant 3$ it is always possible to find a finite set of rays that cannot each be assigned the value 0 or 1 in such a way that both the following conditions are met: (i) no two orthogonal rays are both assigned the value 1; and (ii) in any group of n mutually orthogonal rays, not all the rays are assigned the value 0. The BKS theorem is of fundamental importance in connection with interpretations of the quantum theory because it implies that any hidden-variables theory that would assign a definite value to the result of every quantum measurement, and still reproduce the statistical results of the quantum theory, must necessarily be contextual.

Kochen and Specker [2] first exhibited a set of 117 rays in three dimensions that proved their theorem. However, in recent years many new and simpler proofs of the BKS theorem have been given in three, four and higher dimensions. Peres [4, 5] discovered very symmetrical configurations of rays in three and four dimensions that prove the theorem. Mermin [6] exhibited arrays of observables in four and eight dimensions that lead to elegant (and almost trivial) proofs of the theorem. And Penrose [7] proposed a symmetrical configuration of rays in four dimensions (the 'Penrose dodecahedron') that can also be used to prove the theorem. Proofs of the BKS theorem in dimensions higher than four have been given by Kernaghan and Peres [8] and by DiVincenzo and Peres [9]. Other work related to various aspects of the BKS theorem may be found in [10–13]. Good surveys of recent work on the BKS theorem and its relation to the more famous Bell's theorem may be found in the book by Peres [5] and in the review articles by Mermin [6] and by Brown [14].

The purpose of this paper is to present two configurations of rays in four dimensions, based on the geometry of the 600-cell and the 120-cell, that vindicate the BKS theorem. The 600-cell and 120-cell are the two most complex of the four-dimensional regular polytopes and furnish us with configurations of 60 and 300 rays, respectively, that prove the theorem. Both these configurations possess a high degree of symmetry that make the proof of the theorem quite straightforward.

An attractive feature of many recent proofs of the BKS theorem is the high symmetry of the configurations of rays involved. Thus, Peres' 33-ray configuration [4] in three

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Table 1. The 60 rays associated with antipodal pairs of vertices of the 600-cell. Note: $t = \frac{1+\sqrt{5}}{2}$ and $k = t^{-1}$.

60 rays corresponding to the vertices of the 600-cell							
1	1 0 0 0	16	t −1 −k 0	31	−1 −k t 0	46	−k 0 t −1
2	0 1 0 0	17	t k 0 −1	32	−1 k −t 0	47	k 0 −t −1
3	0 0 1 0	18	−t k 0 −1	33	−1 0 k t	48	k 0 t 1
4	0 0 0 1	19	t −k 0 −1	34	1 0 k t	49	0 t k −1
5	1 1 1 1	20	t k 0 1	35	−1 0 −k t	50	0 −t k −1
6	−1 1 1 1	21	t 0 −1 k	36	−1 0 k −t	51	0 t −k −1
7	1 −1 1 1	22	−t 0 −1 k	37	k t −1 0	52	0 t k 1
8	1 1 −1 1	23	t 0 1 k	38	−k t −1 0	53	0 −1 t k
9	1 1 1 −1	24	t 0 −1 −k	39	k −t −1 0	54	0 1 t k
10	−1 −1 1 1	25	−1 t 0 k	40	k t 1 0	55	0 −1 −t k
11	−1 1 −1 1	26	1 t 0 k	41	k −1 0 t	56	0 −1 t −k
12	−1 1 1 −1	27	−1 −t 0 k	42	−k −1 0 t	57	0 k −1 t
13	t −1 k 0	28	−1 t 0 −k	43	k 1 0 t	58	0 −k −1 t
14	−t −1 k 0	29	−1 k t 0	44	k −1 0 −t	59	0 k 1 t
15	t 1 k 0	30	1 k t 0	45	k 0 t −1	60	0 k −1 −t

dimensions possesses the symmetries of a cube while Penrose's 40-ray configuration [7] in four dimensions possesses the symmetries of a regular dodecahedron. Peres' 24 rays in four dimensions [4] can be related to two different geometrical objects: on the one hand [15] they are the directions from the centre of a tesseract to its 16 vertices, the centres of its 8 cubical cells, and the centres of its 24 square faces (with oppositely directed rays counted only once); and, on the other hand [5], they correspond to the vertices of a pair of dual 24-cells (with 12 rays originating in each 24-cell). These connections between Peres' rays and the simpler four-dimensional regular polytopes made us wonder if the two most complicated of these polytopes might also furnish us with rays proving the BKS theorem. The answer to this question is in the affirmative, as we demonstrate in this paper.

Before proceeding to our demonstration, we say a few words about the 600- and 120-cells. The 600-cell is a symmetrical polytope having 600 tetrahedra for its bounding cells and 120 vertices arranged on the surface of a 4-dimensional sphere. It dual, the 120-cell, has 120 dodecahedra for its bounding cells and 600 vertices arranged on the surface of a 4-dimensional sphere. The vertices of both solids come in antipodal pairs, so by joining the centre of either solid to its vertices we obtain 60 rays (in the case of the 600-cell) or 300 rays (in the case of the 120-cell) with which to try and prove the BKS theorem. The (real) coordinates of the rays for both the 60- and 300-ray sets constitute the only knowledge about these polytopes needed for the argument of this paper. A more detailed account of these polytopes, including their connections with other symmetrical figures, may be found in the text by Coxeter [16].

We first demonstrate how the 60 rays of the 600-cell establish the BKS theorem. The demonstration is contained in tables 1–3 which, together with their captions, convey the entire argument. We merely add some amplifying remarks here. Table 1 shows the coordinates of the 60 rays, based on the vertices of the 600-cell given by Coxeter [16]. Table 2 lists the 75 orthogonal tetrads formed by these rays. It is quite easy to deduce that there must be exactly 75 tetrads. First note that any ray, say ray 1, is orthogonal to exactly 15 other rays and occurs in 5 tetrads whose remaining members are just these rays. Then note that the property just stated holds for all the rays because the symmetry group of the

Table 2. The 75 orthogonal tetrads formed by the 60 rays of the 600-cell (rays are referred to by the numerical labels introduced in table 1). Each ray occurs in exactly 5 tetrads whose remaining 15 members are the only rays it is orthogonal to.

75 tetrads formed by the 60 rays of the 600-cell				
1 2 3 4	4 15 29 38	7 26 29 35	11 21 40 60	17 36 38 54
1 49 56 59	4 16 30 37	8 13 42 54	11 27 29 34	18 33 40 56
1 50 55 60	5 10 11 12	8 19 48 49	12 15 44 53	18 35 37 54
1 51 54 57	5 16 42 56	8 22 39 60	12 17 48 50	19 34 37 55
1 52 53 58	5 19 46 51	8 28 30 33	12 22 37 59	19 36 40 53
2 21 35 48	5 24 38 58	9 16 43 53	12 26 32 33	20 33 38 55
2 22 36 47	5 28 31 35	9 18 47 50	13 26 47 58	20 35 39 53
2 23 33 46	6 7 8 9	9 21 38 59	13 27 46 59	21 30 44 51
2 24 34 45	6 15 41 56	9 25 31 34	14 25 48 58	21 32 42 52
3 17 28 43	6 20 45 51	10 13 43 55	14 28 45 59	22 29 43 51
3 18 27 44	6 23 37 58	10 18 45 52	15 25 46 60	22 31 41 52
3 19 26 41	6 27 32 36	10 23 39 57	15 28 47 57	23 29 42 50
3 20 25 42	7 14 44 55	10 25 30 36	16 26 45 60	23 31 44 49
4 13 31 40	7 17 46 52	11 14 41 54	16 27 48 57	24 30 41 50
4 14 32 39	7 24 40 57	11 20 47 49	17 34 39 56	24 32 43 49

polytope is transitive on all the rays†. Therefore the total number of tetrads is $60 \times \frac{5}{4} = 75$. All the tetrads may in fact be obtained from a single one by applying the symmetries of the polytope. However, we simply used a computer program to generate the tetrads in table 2 from the ray coordinates in table 1.

In order to prove the BKS theorem we must show that it is impossible to assign a 0 or 1 to each of the 60 rays in such a way that there is exactly one 1 and three 0s in each of the tetrads of table 2. Following Peres [4], we equate assigning a 1 or a 0 to a ray with colouring it green or red, respectively. Then the task is to show that the rays cannot each be coloured red or green in such a way that there is exactly one green ray (and three red rays) in each of the 75 tetrads. The 'proof-tree' in table 3 demonstrates the impossibility of this colouring and so proves the BKS theorem. Note that the only input required for this proof is the tetrad table given in table 2.

The BKS proof for the 300 rays of the 120-cell follows along similar lines. In this case there are $675 (= 300 \times \frac{9}{4})$ tetrads, but the proof-tree establishing the non-colourability is not particularly longer than that of table 3. We do not list the analogues of tables 1–3 for this polytope, as the analogues of tables 1 and 2 would fill several pages.

It is interesting to compare the above two configurations with Peres' 24-ray set and Penrose's 40-ray set in order to bring out some similarities and differences. Some results of such a comparison are summarized in table 4. First we note that Peres' rays and ours have purely real components whereas Penrose's rays are essentially complex (i.e. they cannot all be rendered real by any choice of basis). This shows that our sets of rays do not contain Penrose's as a subset. It is also obvious, from an inspection of table 1 (and the analogous table for the 120-cell), that neither of these sets contains all of Peres's 24 rays as a subset. In fact, the 600-cell contains 12 of Peres' rays (the ones corresponding to the vertices of a 24-cell) while the 120-cell contains the remaining 12 rays (the ones corresponding to the vertices of the dual 24-cell). Thus the non-colourability of our configurations is not trivially implied by that of either the Peres or Penrose configurations.

† This simply means that it is possible to send any ray into any other ray by means of a suitable symmetry operation of the polytope.

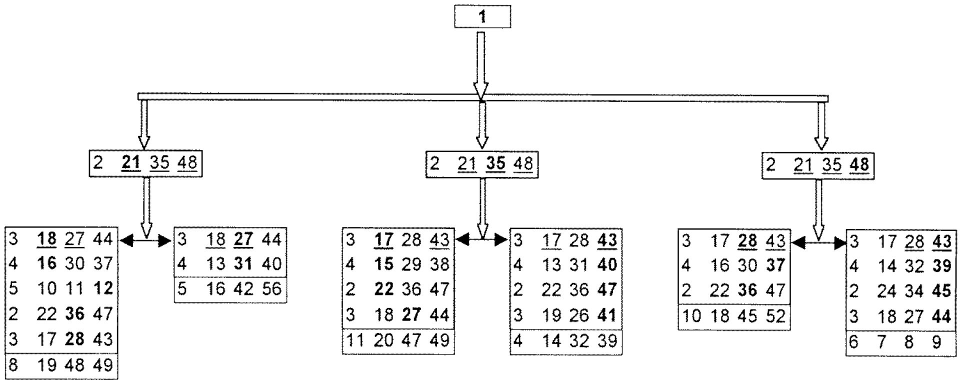

Table 3. Non-colourability of the 60 rays of the 600-cell. One begins by assuming that a colouring exists and shows that this leads to a contradiction. The argument is presented in the form of a 'proof-tree' that proceeds from the root, at the top, down through the various branches to the contradictions at the bottom. Rays coloured green are shown in boldface and those coloured red (all the rest) in ordinary type. Any ray that is underlined has its colour assigned to it as a matter of choice while any ray that is not underlined has its colour forced. A forced red ray always results from a (forced or unforced) green ray in an earlier step, while a forced green ray is always forced by its three red companions in the same step. The (attempted) colouring begins at the root with ray 1 chosen green; there is no arbitrariness about this choice because some ray has to be coloured green and all the rays are equivalent to each other by symmetry (i.e. any ray can be made to pass into any other ray by a suitable symmetry operation of the polytope). The argument then proceeds down the different branches of the proof-tree, corresponding to the various (mutually exclusive and exhaustive) choices of green rays at the second and third steps. All six branches of the proof-tree eventually lead to completely red tetrads, thereby establishing the non-colourability of these 60 rays.

Table 4. Comparisons between some four-dimensional configurations that prove the Kochen–Specker theorem.

4D configurations of rays that prove the K–S theorem				
Authors	Number of rays	Number of tetrads	Symmetry	Critical set
Peres [4]	24	24	Hypercubic	18
Penrose [7]	40	40	Dodecahedron	28
This work	60	75	600-cell	44
This work	300	675	120-cell	89

All four configurations in table 4 enjoy a high degree of symmetry characteristic of some regular solid in three or four dimensions. Our configurations have the additional feature that their underlying symmetry groups are transitive on all the rays, whereas this is not true of the other two configurations: Peres's rays divide into two sets of 12 (corresponding to reciprocal 24-cells) and Penrose's rays into two sets of 20 (the 'explicit' and 'implicit' rays) that cannot be made to pass into each other by symmetry operations of the underlying polytope. A striking feature of both the Peres and Penrose configurations is that the number of rays is exactly equal to the number of tetrads. For our configurations, the tetrads outnumber the rays; this turns out to be an advantageous feature, in view of the large numbers of rays involved, since it makes the colouring process terminate more rapidly than it otherwise would.

A common feature of all four configurations is that all orthogonalities between pairs of rays are represented in the tetrad table for the configuration. Thus the tetrad table provides all the information (and in fact the only relevant information) that has to be kept in view during the colouring process. This is not the case, for example, with two prominent three-dimensional configurations: the 33 rays of Peres [4] and the 31 rays of Kochen and Conway [17]; in both these cases the list of orthogonal triads must be supplemented by an additional list of pair orthogonalities in order to provide all the input required for the non-colouring proof.

An interesting concept connected with a non-colourable configuration is that of a 'critical set'. Following Zimba and Penrose [7] we can term a configuration 'critical' if it is uncolourable but every proper subset of it (obtained by the deletion of one or more rays) is colourable. It is interesting to ask if the four configurations listed in table 4 are critical or not. It is known that Peres' 24 rays are not critical: Kernaghan and Peres [8] pointed out a subset of 20 rays that is critical and later Cabello *et al* [15] pointed out many subsets of 18 that are critical. Aside from their criticality these subsets possess the nice feature that they provide very transparent ('look-see') proofs of the BKS theorem. Zimba and Penrose [7] pointed out that their 40 rays are not critical but contain several subsets of 28 that are. In view of these precedents, we investigated whether our own sets of 60 and 300 rays were critical and found that they were not. After a computer search we found that the 60 rays contained a critical subset of 44 rays† and the 300 rays a critical subset of 89 rays. While the criticality of these subsets is not in doubt, we do not claim that they are the smallest such subsets. Perhaps a clever use of symmetry or a more diligent search will yield still smaller critical sets.

All four configurations in table 4 can be used to prove Bell's nonlocality theorem [18] if used in conjunction with a pair of spin-$\frac{3}{2}$ particles in a singlet state. The way to do this was indicated by Zimba and Penrose [7] for their 40 rays, but a similar method works for the other configurations as well. The idea is to use the anticorrelation between the particles in a singlet state to justify the assumption of noncontextuality made in colouring the rays of either particle. While this artifice is formally acceptable, it is open to the practical objection that projection measurements corresponding to arbitrary states of a spin-$\frac{3}{2}$ particle are not easy to carry out. Zimba and Penrose [7] sidestepped this objection by showing that projection measurements corresponding to only 20 of their rays (the so-called 'explicit rays') needed to be made; these measurements are easily performed with the aid of suitably oriented Stern–Gerlach magnets and detectors.

In conclusion, we have shown how configurations of rays based on the 600- and 120-cells can be used to illustrate the validity of the BKS theorem in a Hilbert space of dimension four. While these demonstrations add little that is essentially new to the many proofs that have preceeded them, we think it a splendid joke that these delightful and quixotic objects can be enlisted as combatants in repulsing an attack on the quantum theory.

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† A critical set of 44 rays may be obtained by deleting rays 1–8, 10, 11, 13, 14, 19, 23, 29 and 58.

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